High angular resolution diffusion MRI reconstruction through denoising and reweighted $\ell_1$-minimization
**FOD RECOVERY VIA SPHERICAL DECONVOLUTION**

*Spherical Deconvolution* methods assume the Orientation Distribution Function can be expressed as the convolution of a kernel with the Fiber Orientation Distribution:

\[ \text{FOD} = \text{ODF} \ast \text{KERNEL} \]

- **ODF**: Probability of diffusion along a given direction
- **KERNEL**: Response generated by a single fiber
- **FOD**: Indicates the **orientation** and the **volume fraction** of the fibers in a voxel

✓ non-negative
✓ sparse
FOD RECOVERY VIA SPHERICAL DECONVOLUTION

*Spherical Deconvolution* methods assume the Orientation Distribution Function (ODF) can be expressed as the convolution of a kernel with the Fiber Orientation Distribution (FOD):

\[
\text{SIGNAL} \quad \text{ODF} \quad \text{KERNEL} \quad \text{FOD}
\]

- ODF: Probability of diffusion along a given direction
- KERNEL: Response generated by a single fiber
- FOD: indicates the **orientation** and the **volume fraction** of the fibers in a voxel

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FOD RECOVERY VIA SPHERICAL DECONVOLUTION

The intra-voxel structure recovery problem can be expressed in terms of the following linear formulation (Jian and Vemuri, 2007):

\[ y = \Phi x + \eta \]

- \( y \) is the acquired diffusion MRI data and \( x \) is the FOD.
- \( \Phi \) is the sensing basis or dictionary typically estimated from the data.
- \( \eta \) represents acquisition noise.

State-of-the-Art methods solve the problem for FOD reconstruction using **convex optimization**:

\[
\begin{align*}
\min_{x \geq 0} \Vert x \Vert_1 & \quad \text{s.t.} \quad \Vert \Phi x - y \Vert_2 \leq \epsilon \\
\min_{x \geq 0} \Vert \Phi x - y \Vert_2^2 + \beta \Vert x \Vert_1 & \quad \text{however...}
\end{align*}
\]

(Ramirez-Manzanares et al., 2007; Landman et al., 2012; Jian and Vemuri, 2007; Pu et al., 2011)
\( \ell_1 \) norm prior is inconsistent!!

\[
\begin{align*}
\min_{x \geq 0} & \quad \|x\|_1 \\
\text{s.t.} & \quad \|\Phi x - y\|_2 \leq \epsilon
\end{align*}
\]

\[
\min_{x \geq 0} \quad \|\Phi x - y\|_2^2 + \beta \|x\|_1
\]

(Ramirez-Manzanares et al., 2007; Jian and Vemuri, 2007)

(Landman et al., 2012; Pu et al., 2011)

Trying to minimize \( \ell_1 \) norm while we have the physical constraint

\[
\|x\|_1 = \sum_i |x_i| = 1
\]

sum of the volume fractions of the compartments inside a voxel
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\begin{align*}
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\[
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\end{align*}
\]

Trying to minimize \( \ell_1 \) norm while we have the physical constraint \( \|x\|_1 = \sum_i |x_i| = 1 \)

sum of the volume fractions of the compartments inside a voxel

Original formulation: \( \ell_0 \)-norm as a prior to minimize sparsity

\[
\begin{align*}
\min_{x \geq 0} & \quad \|\Phi x - y\|_2^2 \\
\text{s.t.} & \quad \|x\|_0 \leq k
\end{align*}
\]

sparsity term
\textbf{FOD RECOVERY VIA SPHERICAL DECONVOLUTION}

\textbf{\ell_1 norm prior is inconsistent!!}

\[
\min_{x \geq 0} \|x\|_1 \quad \text{s.t.} \quad \|\Phi x - y\|_2 \leq \epsilon
\]

\[
\min_{x \geq 0} \|\Phi x - y\|_2^2 + \beta \|x\|_1
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(Ramirez-Manzanares et al., 2007; Jian and Vemuri, 2007)

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Trying to minimize \(\ell_1\) norm while we have the physical constraint

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sum of the volume fractions of the compartments inside a voxel

Original formulation: \(\ell_0\)-norm as a prior to minimize sparsity

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\min_{x \geq 0} \|\Phi x - y\|_2^2 \quad \text{s.t.} \quad \|x\|_0 \leq k
\]

\(\ell_0\) problems are non convex!

sparsity term
Reweighted constrained $\ell_1$ minimization (Candes et al., 2008):

$$\min_{x \geq 0} \left\| \Phi x - y \right\|_2^2 \quad \text{s.t.} \quad \|x\|_1^w \leq k$$

where

$$\|x\|_1^w = \sum_i w_i |x_i|$$
Reweighted constrained $\ell_1$ minimization (Candes et al., 2008):

$$
\min_{x \geq 0} \| \Phi x - y \|^2_2 \quad \text{s.t.} \quad \| x \|^w_1 \leq k
$$

Solving a sequence of these weighted problems with $w_i^{(t)} \approx 1/x_i^{(t-1)}$

✓ We approximate the $\ell_0$ minimization
✓ We solve the $\ell_1$ inconsistency

Effect of different regularization priors

Impact on reconstructions
The initial data have been denoised using a **total variation prior** for each $q$-point:

$$\min_{\hat{y}^Z \geq 0} \| \hat{y}^Z \|_{TV} \quad \text{s.t.} \quad \| y^Z - \hat{y}^Z \|_2 < \epsilon, \quad \text{for all } Z.$$ 

The data is denoised **slice by slice** (2D-TV)!
IMPROVEMENTS: SPATIAL REGULARIZATION

1. Denoise the whole 3D-volume for each \( q \)-point at a time (using TV-3D):

\[
\min_{\hat{y} \geq 0} \| \hat{y} \|_{TV} \quad \text{s.t.} \quad \| y - \hat{y} \|_2 < \epsilon,
\]

where \( y \) is a concatenation of all \( y^Z \).

2. Formulate a **global problem** including **spatial regularization**:

\[
\min_{X \in \mathbb{R}^{n \times N^3}} \| X \|_{0,1} + \beta \| X \|_{TV} \quad \text{s.t.} \quad \| \Phi X - Y \|_2 \leq \epsilon
\]

solve for all the voxels at the same time, instead of each voxel sequentially

to exploit spatial coherence in neighboring voxels